9.2

4 (c)

$$e^{-\ln n} = \frac{1}{e^{\ln n}} = \frac{1}{n}$$

So $\left\{ \sum_{n=1}^{m} e^{-\ln n} \right\}_{m=1}^{\infty}$ is harmonic series, which is divergent.

(f) $$\frac{(n+1)!\, e^{-(n+1)^2}}{n!\, e^{-n^2}} = (n+1) e^{-2n-1} = \frac{n+1}{e^{2n+1}}$$

Since $\lim_{n \to \infty} \frac{n+1}{e^{2n+1}} = 0$

$\Rightarrow \exists N \in \mathbb{N}$ s.t. $\forall n > N$, we have $\frac{n+1}{e^{2n+1}} < \frac{1}{2}$

Then by Ratio Test we have

$\left\{ \sum_{n=1}^{m} \frac{n+1}{e^{2n+1}} \right\}_{m=1}^{\infty}$ is convergent.

7

(b)

$$\frac{\dfrac{((n+1)!)^2}{(2n+2)!}}{\dfrac{(n!)^2}{(2n)!}} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{(n+1)^2}{(2n+2)(2n+2-1)} = \frac{1}{4}\,\frac{1}{1-\frac{1}{4n+4}}$$

Since $\lim_{n\to\infty} \frac{1}{4}\,\frac{1}{1-\frac{1}{4n+4}} = \frac{1}{4}$

Then $\exists\, N\in\mathbb{N}$ s.t. $\forall\, n>N$, $\frac{1}{4}\,\frac{1}{1-\frac{1}{4n+4}} < \frac{1}{3}$

By Ratio Test,

we have $\left\{\sum_{n=1}^{m} \frac{(n!)^2}{2n!}\right\}_{m=1}^{\infty}$ is convergent

(d)

$$\frac{\dfrac{2\cdot 4\cdot \cdots \cdot 2n\cdot(2n+2)}{5\cdot 7\cdot \cdots (2n+3)(2n+5)}}{\dfrac{2\cdot 4 \cdots 2n}{5\cdot 7\cdot \cdots (2n+3)}} = \frac{2n+2}{2n+5} = 1-\frac{3}{2n+5}$$

Then we have $\lim_{n\to\infty} n\left(1-\dfrac{\frac{2\cdot 4\cdots 2n\cdot(2n+2)}{5\cdot 7\cdots(2n+3)(2n+5)}}{\frac{2\cdot 4\cdots 2n}{5\cdot 7\cdots(2n+3)}}\right) = \lim_{n\to\infty} n\left(\frac{3}{2n+5}\right)$

$$= \lim_{n\to\infty} \frac{3n}{2n+5}$$

$$= \frac{3}{2} > 1$$

By Coro. 9.2.9

we have $\left\{\sum_{n=1}^{m} \frac{2\cdot 4\cdots 2n}{5\cdot 7\cdots (2n+3)}\right\}_{m=1}^{\infty}$ is convergent

9.3

1. (c) let $a_n = \frac{(-1)^{n+1} n}{n+2}$

Since $|a_n| = \frac{n}{n+2} \geq \frac{1}{3}$ $\forall n \in \mathbb{N}, n \geq 1$

$\Rightarrow$ it is not true that $\lim_{n\to\infty} a_n = 0$

By n-th term test

we have $\left\{ \sum_{n=1}^{m} a_n \right\}_{m=1}^{\infty}$ is divergent

(d) By Alternating Series test

we only need to check

① $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

② $\frac{\ln n}{n}$ is decreasing for sufficient large n

for ①, by L'hospital rule, we have

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$$

for ②, let $f(x) = \frac{\ln x}{x}$ $f'(x) = \frac{1}{x^2} - \frac{\ln x}{x^2} = \frac{1-\ln x}{x^2}$

for $x \geq e$ $(n \geq 3)$

we have $f'(x) < 0 \Rightarrow f(x)$ is decreasing

$\Rightarrow \left\{ \frac{\ln n}{n} \right\}_{n=3}^{\infty}$ is decreasing

② Since $|b_i| = \frac{\ln n}{n}$

Consider function $f(x) = \frac{\ln x}{x} > 0$ & decreasing on $[e, +\infty)$

$$\int_e^n \frac{\ln x}{x} dx = \int_e^n \ln x \, d\ln x \overset{t=\ln x}{=} \int_1^{\ln n} t\,dt > \ln n - 1$$

is divergent

$\Rightarrow \left\{ \sum_{i=1}^{n} |b_i| \right\}_{n=1}^{\infty}$ is divergent by Integral test

7. By Alternating Series Test

it's sufficient to show that ($\frac{(\ln n)^p}{n^q}$ decreasing for sufficient large n

$$\lim_{n\to\infty} \frac{(\ln n)^p}{n^q} = 0 \qquad \forall\, p, q > 0$$

as $\frac{d}{dx} f(x) = \frac{d}{dx} \frac{(\ln x)^p}{x^q} = \frac{(\ln x)^{p-1}(pq - \ln x)}{q x^{q+1}}$

< 0 for $x > e^{pq}$)

let $n = e^{x_n}$ for some x_n, we have $\{x_i\}_{i=1}^{\infty}$ is increasing

and $\lim_{i\to\infty} x_i = \infty$, we have $\frac{(\ln n)^p}{n^q} = \frac{(\ln e^{x_n})^p}{e^{qx_n}} = \frac{x_n^p}{e^{qx_n}}$

By L'hospital rule,

$$\lim_{x\to\infty} \frac{x^p}{e^{qx}} = \lim_{x\to\infty} \frac{1 \cdot 2 \cdots \lfloor p \rfloor + 1 \cdot x^{p-\lfloor p \rfloor - 1}}{q^{\lfloor p \rfloor + 1} e^{qx}}$$

($\lfloor p \rfloor$ means the integer component of p, i.e. $p = \lfloor p \rfloor + \alpha$, $\lfloor p \rfloor \in \mathbb{Z}$, $\alpha \in [0,1)$)

Since $\lim_{x\to\infty} x^{p-\lfloor p \rfloor - 1} = 0$

$\lim_{x\to\infty} e^{qx} = \infty$

we have $\lim_{n\to\infty} \frac{x_n^p}{e^{qx_n}} = 0$, hence $\sum (-1)^n \frac{(\ln n)^p}{n^q}$ is

Convergent

3 (b)

$$b_n = \frac{n^n}{(n+1)^{n+1}} = \frac{1}{n}\frac{n^{n+1}}{(n+1)^{n+1}} = \frac{1}{n}\frac{1}{(1+\frac{1}{n})^{n+1}}$$

Since $\lim_{n\to\infty}(1+\frac{1}{n})^{n+1} = \lim_{n\to\infty}(1+\frac{1}{n})^{n}\cdot(1+\frac{1}{n}) = e$

$\Rightarrow \exists N \in \mathbb{N}$ s.t. $\forall n_0 > N$, $n_0 \in \mathbb{N}$, $\frac{1}{(1+\frac{1}{n})^{n+1}} > \frac{1}{3}$

$$\Rightarrow \sum_{n=1}^{m} b_n \geq \sum_{n=1}^{n_0} b_n + \sum_{n_0+1}^{m} b_n > \sum_{i=n_0+1}^{m} \frac{1}{3i}$$

Since harmonic series is divergent

we have $\{\sum b_n\}$ is divergent.

(c) $C_n = (-1)^n \frac{(n+1)^n}{n^n} = (-1)^n(1+\frac{1}{n})^n$

Since $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$, $\Rightarrow \exists N \in \mathbb{N}$, s.t. $\forall n > N$, we have

$(1+\frac{1}{n})^n > 2$, then by n-th term test,

we have $\sum C_n$ is divergent.

9.4

1. (a) $\sum_{n=m_1}^{m_2} \frac{1}{x^2+n^2} < \sum_{n=m_1}^{m_2} \frac{1}{n^2} < \sum_{n=m_1}^{m_2} \frac{1}{n(n-1)} = \sum_{n=m_1}^{m_2} \frac{1}{n-1} - \frac{1}{n}$

$< \frac{1}{m_1-1}$

$\forall x \in \mathbb{R}$.

$\Rightarrow \forall x \in \mathbb{R}, \forall \varepsilon > 0, \exists N \in \mathbb{N}$, e.g. $N > \frac{1}{\varepsilon}+1$, s.t. $\forall m_2 > m_1 > N$,

we have $\sum_{n=m_1}^{m_2} \frac{1}{x^2+n^2} < \varepsilon$.

$\Rightarrow \sum f_n$ is uniformly convergent

(c)

① Recall the inequality $\sin x \leq x$ for $x \geq 0$

$\Rightarrow \sum_{m_1}^{m_2} \sin\frac{x}{n^2} \leq x \sum_{m_1}^{m_2} \frac{1}{n^2} \leq \frac{x}{m_1-1}$

$\Rightarrow \forall x \in \mathbb{R}, \forall \varepsilon > 0, \exists N \in \mathbb{N}$ e.g. $N > \frac{x}{\varepsilon}+1$ s.t. $\forall m_2 > m_1 > N$

we have $\sum_{n=m_1}^{m_2} \sin\frac{x}{n^2} < \varepsilon$

$\Rightarrow \sum \sin\frac{x}{n^2}$ convergent to some function f.

② Only the other hand, for $\forall n \in \mathbb{N}$, $\exists x = (n+1)^2 \frac{\pi}{2}$ s.t.

$\sup \left| \sum_{i=1}^{n+1} \sin\frac{x}{i^2} - \sum_{i=1}^{n} \sin\frac{x}{i^2} \right| = \sup \left| \sin\frac{x}{(n+1)^2} \right| = 1$

$\Rightarrow \sum \sin\frac{x}{n^2}$ do not converge uniformly.

6 (a)

Since $\left(\frac{1}{n^n}\right)^{\frac{1}{n}} = \frac{1}{n}$

we have $\limsup_n \left(\frac{1}{n^n}\right)^{\frac{1}{n}} = \limsup_n \frac{1}{n} = 0$

$\Rightarrow$ the convergence radius is ∞

(e) The convergence radius of sequence $\sum \frac{(n!)^2}{2n!} x^n$

is given by $\lim_{n\to\infty} \frac{\frac{(n!)^2}{2n!}}{\frac{(n+1)!^2}{2(n+1)!}} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)^2} = 4$

$\Rightarrow$ convergence radius $R = 4$